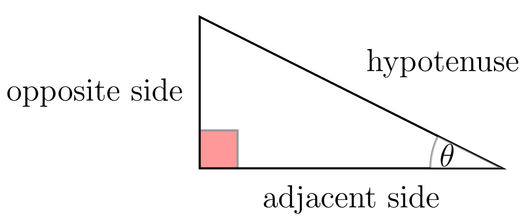
Facts about Trigonometric Substitution	**Explanation**
How is Trigonometric Substitution Used?	Trigonometric Substitution (Trig. Sub.) helps simplify a radical or an expression. It is useful if the integrand contains a sum or a difference of squares, often under a radical.
Expression — Trig. Sub. — Identity $\sqrt{a^2 - x^2}$ — $x = asin(\theta)$ — $sin^2(\theta) + cos^2(\theta) = 1$ $\sqrt{x^2 - a^2}$ — $x = asec(\theta)$ — $tan^2(\theta) + 1 = sec^2(\theta)$ $\sqrt{x^2 + a^2}$ — $x = atan(\theta)$ — $tan^2(\theta) + 1 = sec^2(\theta)$	A summary of trigonometric substitutions in which the expression considered, trigonometric substitution used, and pythagorean identity applied are in the left column, middle column, and right column respectively.
Review of the Relationship between Trigonometric Functions and Right Triangles 	$sin(\theta) = \frac{\text{opposite side}}{\text{hypotenuse}}$ $csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite side}}$ $cos(\theta) = \frac{\text{adjacent side}}{\text{hypotenuse}}$ $sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent side}}$ $tan(\theta) = \frac{\text{opposite side}}{\text{adjacent side}}$ $cot(\theta) = \frac{\text{adjacent side}}{\text{opposite side}}$

1. We will be applying Trigonometric Substitution in order to evaluate $\int \frac{1}{\sqrt{1-x^2}}\, dx$ through the following parts.

   (a) Determine the appropriate Trigonometric Substitution to use out of the following: $x = asin(\theta), x = atan(\theta)$, or $x = asec(\theta)$. Afterwards, compute dx.

   Solution:
   $$x = (1)sin(\theta) = sin(\theta) \qquad dx = cos(\theta)d\theta$$

   (b) Substitute x and dx and simplify as much as possible, using trigonometric identities to get rid of the square root.

   Solution:
   $$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\, dx &= \int \frac{cos(\theta)}{\sqrt{1-sin^2(\theta)}}\, d\theta \\ &= \int \frac{cos(\theta)}{\sqrt{cos^2(\theta)}}\, d\theta \\ &= \int d\theta + C = \theta + C\end{aligned}$$

   (c) Evaluate the integral and change your answer back in terms of x. Could you have evaluated your integral in an easier way?

   Solution:
   Since $x = sin(\theta) \Leftrightarrow arcsin(x) = \theta$, $\int \frac{1}{\sqrt{1-x^2}}\, dx = arcsin(x) + C$. We can simply have evaluated this by noting that $\frac{d}{dx}arcsin(x) = \frac{1}{\sqrt{1-x^2}}$.

2. We will be applying Trigonometric Substitution in order to evaluate $\int \frac{1}{\sqrt{4+x^2}}\, dx$ through the following parts.

   (a) Determine the appropriate Trigonometric Substitution to use out of the following: $x = asin(\theta), x = atan(\theta)$, or $x = asec(\theta)$. Afterwards, compute dx.

   Solution:
   $$x = 2tan(\theta) \qquad dx = 2sec^2(\theta)d\theta$$

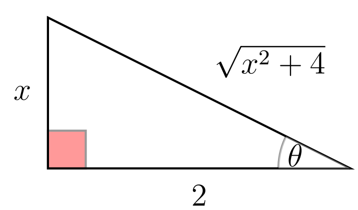
(b) Substitute x and dx and simplify as much as possible, using trigonometric identities to get rid of the square root. Note that $\int sec(\theta)d\theta = ln|sec(\theta) + tan(\theta)|$+C.

Solution:

$$\begin{aligned}\int \frac{1}{\sqrt{4+x^2}}\,dx &= \int \frac{2sec^2(\theta)}{\sqrt{4+4tan^2(\theta)}}\,d\theta \\ &= \int \frac{2sec^2(\theta)}{\sqrt{4}\sqrt{1+tan^2(\theta)}}\,d\theta \\ &= \int \frac{sec^2(\theta)}{\sqrt{sec^2(\theta)}}\,d\theta \\ &= \int \frac{sec^2(\theta)}{sec(\theta)}d\theta \\ &= \int sec(\theta)\ d\theta \\ &= ln|sec(\theta)+tan(\theta)|+C\end{aligned}$$

(c) Evaluate the integral and change your answer back in terms of x. This may require the use of a right triangle.

Solution:
First note that $tan(\theta) = \frac{x}{2}$.



$$\int \frac{1}{\sqrt{4+x^2}} = ln\left|sec(\theta)+tan(\theta)\right| + C = ln\left|\frac{2}{\sqrt{x^2+4}} + \frac{x}{2}\right| + C$$

3. We will be applying Trigonometric Substitution in order to evaluate $\int \frac{\sqrt{x^2-3}}{x}\ dx$ through the following parts.

(a) Determine the appropriate Trigonometric Substitution to use out of the following: $x = asin(\theta), x = atan(\theta)$, or $x = asec(\theta)$. Afterwards, compute dx.

Solution:

$$x = \sqrt{3}sec(\theta) \qquad dx = \sqrt{3}sec(\theta)tan(\theta)d\theta$$

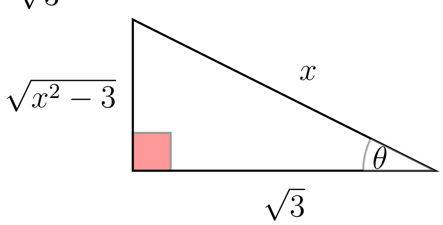
(b) Substitute x and dx and simplify as much as possible, using trigonometric identities to get rid of the square root.

Solution:

$$\begin{aligned}\int \frac{\sqrt{x^2-3}}{x}\ dx &= \int \frac{\sqrt{3sec^2(\theta)-3}}{\sqrt{3}sec(\theta)}\sqrt{3}sec(\theta)tan(\theta)d\theta \\ &= \int \sqrt{3}\sqrt{sec^2(\theta)+1}tan(\theta)\ d\theta \\ &= \int \sqrt{3}tan^2(\theta)\ d\theta \\ &= \int \sqrt{3}(sec^2(\theta)-1)\ d\theta \\ &= \sqrt{3}tan(\theta)-\sqrt{3}\theta+C\end{aligned}$$

(c) Evaluate the integral and change your answer back in terms of x. This may require the use of a right triangle.

Solution:
First note that $sec(\theta) = \dfrac{x}{\sqrt{3}}$.



$$\begin{aligned}\int \frac{\sqrt{x^2-3}}{x} &= \sqrt{3}tan(\theta)-\sqrt{3}\theta+C \\ &= \sqrt{3}\frac{\sqrt{x^2-3}}{\sqrt{3}} - \sqrt{3}sec^{-1}\left(\frac{x}{\sqrt{3}}\right)+C \\ &= \sqrt{x^2-3}-\sqrt{3}sec^{-1}\left(\frac{x}{\sqrt{3}}\right)+C\end{aligned}$$